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The longitudinal shear problem for an array of cracks at the edge of a circular hole in an infinite elastic solid

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SUMMARY

A Mellin-type transform technique reduces the longitudinal shear problem for a set of cracks at the edge of a circular hole in an infinite elastic solid to that of solving a system of integral equations. The stress intensity factors and crack formation energy are calculated. Three special cases are considered in detail and graphical results given.

1. Introduction

We shall consider (Figure 1) an infinite elastic solid containing a circular hole $0 \leqq r \leqq b$, $0 \leqq \theta \leqq 2\pi$ and an array of edge cracks $b \leqq r \leqq bc_i$, $\theta = \beta_i$, $i = 1, 2, 3, \ldots, n$ whose lengths we denote by $a_i = b(c_i - 1)$. The problem we deal with is that of determining the stress intensity factors and the crack formation energy when the cracks and the hole are traction free and the solid is subject to a longitudinal shear load $\sigma_{rz} = T \sin\theta$ and $\sigma_{\theta z} = T\cos\theta$ at infinity.

2. Reduction of the problem to integral equations

Let $\Omega_0 = \{(r, \theta)\colon b < r < \infty,\ 0 \leqq \theta \leqq 2\pi\}$, $\Omega_i = \{(r, \beta_i)\colon b < r \leqq bc_i\}$, $i = 1, 2, \ldots, n$ and $\Omega = \Omega_0 - \bigcup_{i=1}^{n} \Omega_i$. In the longitudinal shear problem the displacement and stress fields are given by the relations

$$u_r = u_\theta = 0,\ u_z = w(r, \theta),\ \sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{r\theta} = 0,$$
$$\sigma_{\theta z} = \frac{\mu}{r}\frac{\partial w}{\partial \theta},\ \sigma_{rz} = \mu\frac{\partial w}{\partial r}, \tag{2.1}$$

where μ is the shear modulus and $w(r, \theta)$ is a solution of the equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0. \tag{2.2}$$

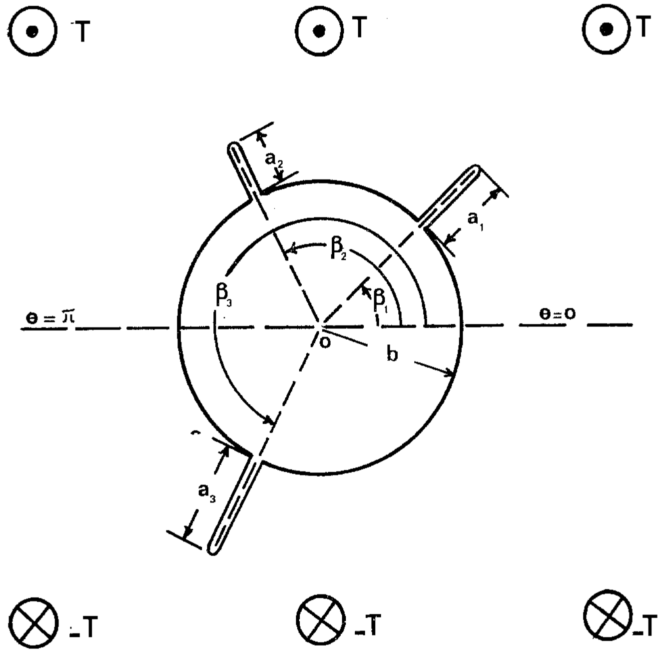

Figure 1.

Therefore, it is sufficient to find a function $w(r, \theta)$ satisfying (2.2) in Ω such that

(1) as $r \to \infty$, $w \to \dfrac{Tr}{\mu} \sin\theta$,

(2) $\dfrac{\partial w}{\partial r}(b, \theta) = 0, \quad 0 \leqq \theta \leqq 2\pi$, and

(3) $\dfrac{\partial w}{\partial \theta}(r, \beta_i +) = \dfrac{\partial w}{\partial \theta}(r, \beta_i -) = 0, \quad b < r < bc_i.$

Let

$$w(r, \theta) = \frac{T}{\mu}\left(r + \frac{b^2}{r}\right) + \sum_{i=1}^{n} \phi_i(r, \theta),$$

where

$$\phi_i(r, \theta) = \sum_{i=1}^{n} H_b^{-1}\left[\frac{A_i(s)}{s} \; \frac{\sin(\theta - \beta_i - \pi)s}{\sin \pi s}; r\right], \tag{2.3}$$

$$0 < \theta - \beta_i < 2\pi, \; \phi_i(r, \theta + 2k\pi) = \phi_i(r, \theta),$$

$$A_i(s) = \frac{T}{\mu} \int_b^{bc_i} \frac{P_i(t)}{[(bc_i - t)(t - b)]^{\frac{1}{2}}} (b^{2s} t^{-s} - t^s)\, dt, \tag{2.4}$$

$i = 1, 2, 3, \ldots, n$, $|\mathrm{Re}(s)| < 1$ and H_b^{-1} is the inverse of the Mellin-type transform [1]

$$H_b[f(r):s] = \int_b^{\infty} (r^{s-1} + b^{2s} r^{-s-1}) f(r)\, dr. \tag{2.5}$$

$w(r, \theta)$ is a solution of (2.2) in $\Omega[2]$, satisfying conditions (1) and (2), and is such that

$$w(r, \beta_i+) - w(r, \beta_i-) = \begin{cases} \dfrac{2T}{\mu}\displaystyle\int_r^{bc_i} \frac{P_i(t)}{[(bc_i - t)(t - b)]^{\frac{1}{2}}}\, dt, & b < r < bc_i, \\ 0, & bc_i < r < \infty. \end{cases} \quad (2.6)$$

Furthermore

$$\frac{\partial w}{\partial \theta}(r, \theta) = \frac{T}{\mu}\left(r + \frac{b^2}{r}\right)\cos\theta + \sum_{i=1}^{n} H_b^{-1}\left[A_i(s)\frac{\cos(\theta - \beta_i - \pi)s}{\sin \pi s}; r\right]$$

$$= \frac{T}{\mu}\left(r + \frac{b^2}{r}\right)\cos\theta +$$

$$+ \frac{T}{\mu}\sum_{i=1}^{n}\int_b^{bc_i} \frac{P_i(t)}{[(bc_i - t)(t - b)]^{\frac{1}{2}}} H_b^{-1}\left[\frac{(b^{2s}t^{-s} - t^s)\cos(\theta - \beta_i - \pi)s}{\sin \pi s}; r\right] dt, \quad (2.7)$$

and hence (3) will be satisfied if

$$\frac{1}{\pi}\sum_{i=1}^{n}\int_b^{bc_i} \frac{P_i(t)}{[(bc_i - t)(t - b)]^{\frac{1}{2}}} K(r, \beta_j - \beta_i, t)\,dt = -\left(r + \frac{b^2}{r}\right)\cos\beta_j, \quad b < r < bc_j, \quad (2.8)$$

where

$$K(r, \theta, t) = \pi H_b^{-1}\left[\frac{(b^{2s}t^{-s} - t^s)\cos(\theta - \pi)s}{\sin \pi s}; r\right]$$

$$= \frac{1}{2}\left\{\frac{r^2 - t^2}{r^2 - 2rt\cos\theta + t^2} + \frac{b^4 - r^2t^2}{r^2t^2 - 2b^2rt\cos\theta + b^4}\right\}. \quad (2.9)$$

Also, since

$$\frac{\partial w}{\partial r}(b, \beta_i) = 0$$

we have

$$P_i(b) = 0, \qquad i = 1, 2, 3, \ldots, n. \quad (2.10)$$

The problem is now reduced to that of solving the integral equations (2.8) subject to the subsidiary conditions (2.10).

3. The stress intensity factors and crack formation energy

The stress intensity factor $K^{(i)}$ at the crack tip (bc_i, β_i) is defined by the equation

$$K^{(i)} = -\frac{\mu}{2}\lim_{r \to bc_i-} [2(bc_i - r)]^{\frac{1}{2}}\frac{\partial}{\partial r}[w(r, \beta_i+) - w(r, \beta_i-)] \quad (3.1)$$

and so, by (2.6),

$$K^{(i)} = \frac{T\sqrt{2}\, P_i(bc_i)}{[b(c_i - 1)]^{\frac{1}{2}}}. \quad (3.2)$$

Similarly the crack formation energy W is defined by

$$W = \tfrac{1}{2} \sum_{i=1}^{n} \int_{b}^{bc_i} \sigma_{\theta z}^{(i)}(r)[w(r, \beta_i+) - w(r, \beta_i-)]\,dr, \tag{3.3}$$

where $\sigma_{\theta z}^{(i)}(r)$ is the shear stress on the line $\theta = \beta_i$ in the absence of the crack. Therefore, by (2.6) and (2.7),

$$W = \frac{T^2}{\mu} \sum_{i=1}^{n} \cos\beta_i \int_{b}^{bc_i} \left(t - \frac{b^2}{t}\right) \frac{P_i(t)\,dt}{[(bc_i - t)(t - b)]^{\frac{1}{2}}}. \tag{3.4}$$

Now [3]

$$K_0^{(i)} = T[b(c_i - 1)]^{\frac{1}{2}} \tag{3.5}$$

and

$$W_i = \frac{\pi T^2 b^2}{4\mu} (c_i - 1)^2 \tag{3.6}$$

are the stress intensity factor and crack formation energy respectively of an edge crack of length $b(c_i - 1)$ in an infinite elastic solid subject to a uniform longitudinal shear T parallel to the crack faces.

Therefore, if $W_0 = \sum_{i=1}^{n} W_i$ we have

$$\frac{K^{(i)}}{K_0^{(i)}} = \frac{\sqrt{2}}{b(c_i - 1)} P_i(bc_i) \tag{3.7}$$

and

$$\frac{W}{W_0} = \frac{4}{\pi b^2 \sum_{i=1}^{n} (c_i - 1)^2} \sum_{i=1}^{n} \cos\beta_i \int_{b}^{bc_i} \left(t - \frac{b^2}{t}\right) \frac{P_i(t)}{[(bc_i - t)(t - b)]^{\frac{1}{2}}}\,dt. \tag{3.8}$$

4. Special cases

Case (a). The first case we consider is that of two cracks of equal length defined by the relations $b \leqq r \leqq bc$ and $\theta = \beta$ or $-\beta$. In this case $P_1(r) = P_2(r)$ and by setting $\tau = t/b$, $\rho = r/b$ and $Q(\tau) = b^{-1} P_1(b\tau)$, (2.8) and (2.10) become

$$\frac{1}{\pi} \int_{1}^{c} \frac{Q(\tau)}{[(c - \tau)(\tau - 1)]^{\frac{1}{2}}} K_1(\rho, \beta, \tau)\,d\tau = -\cos\beta\,(\rho + \rho^{-1}), \quad 1 < \rho < c \tag{4.1}$$

and

$$Q(1) = 0 \tag{4.2}$$

where

$$K_1(\rho, \beta, \tau) = \frac{\tau}{\rho - \tau} + \frac{1}{1 - \rho\tau} + \frac{\rho^2 - \tau^2}{2(\rho^2 - 2\rho\tau\cos 2\beta + \tau^2)} + \frac{1 - \rho^2\tau^2}{2(\rho^2\tau^2 - 2\rho\tau\cos 2\beta + 1)}. \tag{4.3}$$

Using the method of Erdogan and Gupta [4], (4.1) and (4.2) are approximated by the linear algebraic system

$$\frac{1}{m}\sum_{k=1}^{m} Q(t_k)K(r_i, \beta, t_k) = -\cos\beta\,(r_i + r_i^{-1}), \qquad i = 1, 2, \ldots, m-1,$$
$$\frac{1}{m}\sum_{k=1}^{m} (-1)^k \left(\frac{1-x_k}{1+x_k}\right)^{\frac{1}{2}} Q(t_k) = 0, \tag{4.4}$$

where $x_k = \cos[(2k-1)\pi/2m]$, $t_k = \frac{1}{2}(c-1)x_k + \frac{1}{2}(c+1)$, $k = 1, 2, \ldots, m$, and $r_i = = \frac{1}{2}(c-1)\cos(i\pi/m) + \frac{1}{2}(c+1)$, $i = 1, 2, \ldots, m-1$. Having solved these equations for the unknowns $Q(t_k)$, K/K_0 and W/W_0 are calculated from the formulae

$$\frac{K}{K_0} = \frac{\sqrt{2}}{m(c-1)}\sum_{k=1}^{m} (-1)^{k-1}\left(\frac{1+x_k}{1-x_k}\right)^{\frac{1}{2}} Q(t_k) \tag{4.5}$$

and

$$\frac{W}{W_0} = \frac{4\cos\beta}{m(c-1)^2}\sum_{k=1}^{m} (t_k - t_k^{-1})Q(t_k). \tag{4.6}$$

The results of these calculations are given graphically in Figures 2 and 3 which show respectively the variation of K/K_0 and W/W_0 with $a/b = (c-1)$ for several values of β.

Case (*b*). The next case we consider also involves two cracks of equal length, here defined by the relations $b \leqq r \leqq bc$ and $\theta = \frac{1}{2}\pi + \beta$ or $\theta = \frac{1}{2}\pi - \beta$. In this case $P_1(r) = -P_2(r)$ and so, by setting $\tau = t/b$, $\rho = r/b$ and $Q(\tau) = b^{-1}P_1(b\tau)$, (2.8) and (2.10) become

$$\frac{1}{\pi}\int_1^c \frac{Q(\tau)}{[(c-\tau)(\tau-1)]^{\frac{1}{2}}} K_2(\rho, \beta, \tau)\,d\tau = -\sin\beta\,(\rho + \rho^{-1}), \qquad 1 < \rho < c \tag{4.7}$$

and

$$Q(1) = 0 \tag{4.8}$$

where

$$K_2(\rho, \beta, \tau) = \frac{\tau}{\rho - \tau} + \frac{1}{1 - \rho\tau} + \frac{\tau^2 - \rho^2}{2(\rho^2 - 2\rho\tau\cos 2\beta + \tau^2)}$$
$$+ \frac{\rho^2\tau^2 - 1}{2(\rho^2\tau^2 - 2\rho\tau\cos 2\beta + 1)}. \tag{4.9}$$

Here again, the problem is solved by the method of Erdogan and Gupta and the results for K/K_0 and W/W_0 are shown graphically in Figures 4 and 5 respectively.

Case (*c*). The last case we consider is that of two unequal cracks on the same diameter, defined by the relations $b \leqq r \leqq bc_i$ and $\theta = (i-1)\pi$, $i = 1, 2$. Setting $\tau = t/b$, $\rho = r/b$, $Q_1(\tau) = b^{-1}P_1(b\tau)$ and $Q_2(\tau) = b^{-1}P_2(-b\tau)$, (2.8) and (2.10) become

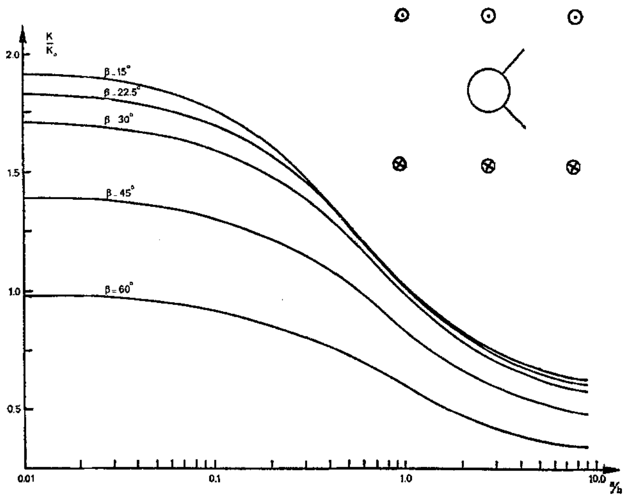

Figure 2. The variation of K/K_0 with $a/b = (c-1)$ for several values of β.

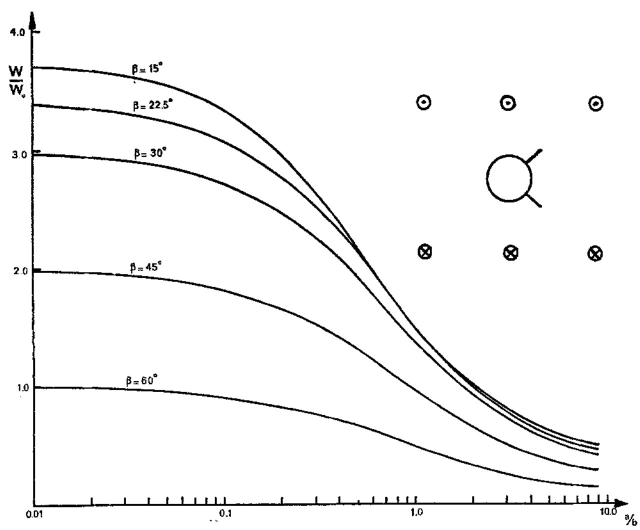

Figure 3. The variation of W/W_0 with $a/b = (c-1)$ for several values of β.

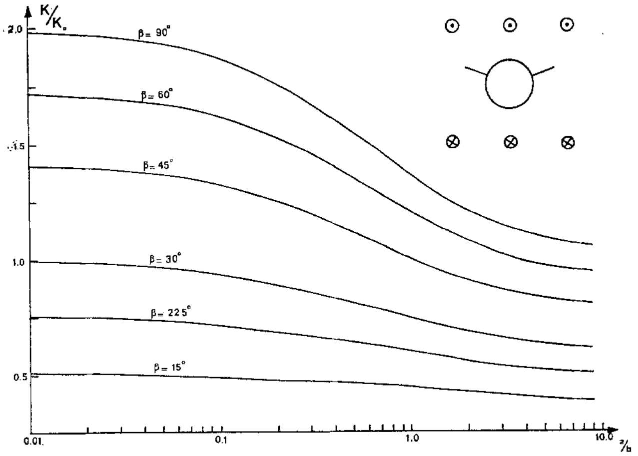

Figure 4. The variation of K/K_0 with $a/b = (c-1)$ for several values of β.

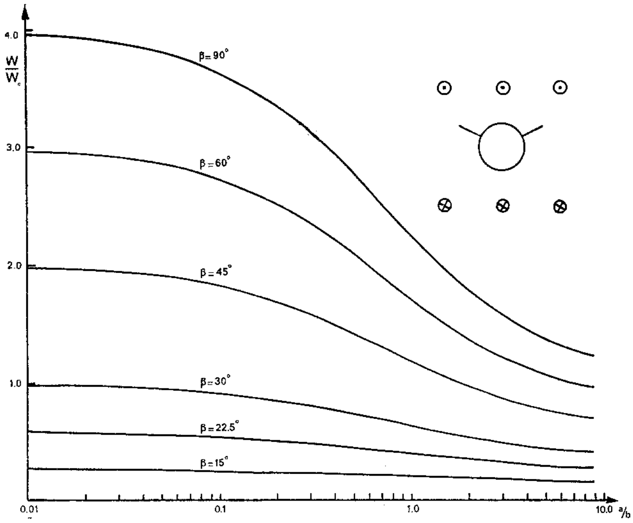

Figure 5. The variation of W/W_0 with $a/b = (c-1)$ for several values of β.

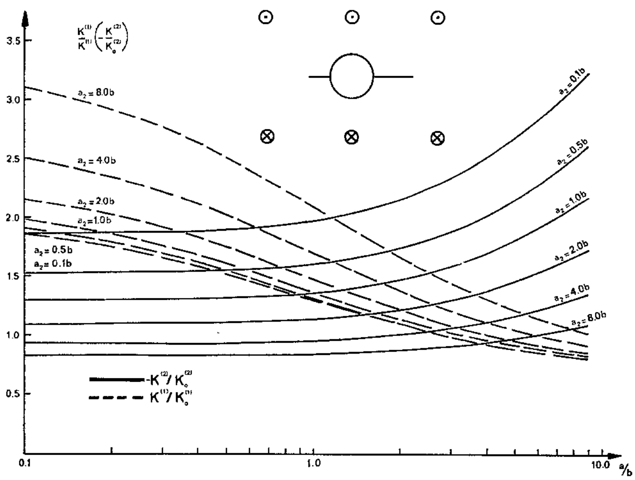

Figure 6. The variation of $K^{(1)}/K_0^{(1)}$ and $-K^{(2)}/K_0^{(2)}$ with a_1/b for several values of a_2/b.

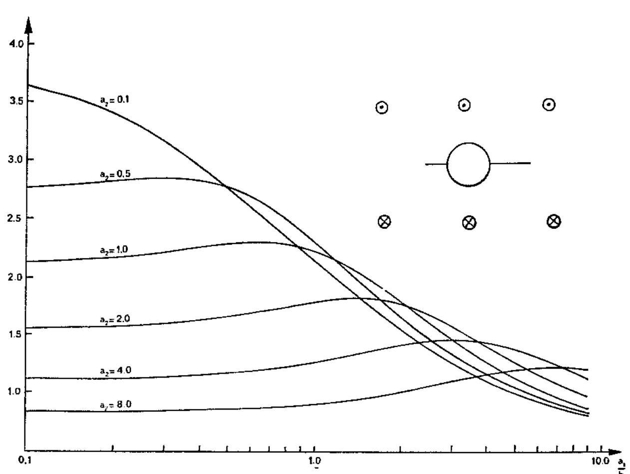

Figure 7. The variation of W/W_0 with a_1/b for several values of a_2/b.

$$\frac{1}{\pi}\int_{-c_2}^{-1}\frac{Q_2(\tau)K_3(\rho,\tau)}{[(-1-\tau)(\tau+c_2)]^{\frac{1}{2}}}\,d\tau+\frac{1}{\pi}\int_{1}^{c_1}\frac{Q_1(\tau)K_3(\rho,\tau)}{[(c_1-\tau)(\tau-1)]^{\frac{1}{2}}}\,d\tau=-(\rho+\rho^{-1}),$$

$$(-c_2<\rho<-1)\bigcup(1<\rho<c_1) \quad (4.10)$$

and

$$Q_2(-1)=Q_1(1)=0 \quad (4.11)$$

where

$$K_3(\rho,\tau)=\frac{\tau}{\rho-\tau}+\frac{1}{1-\rho\tau}. \quad (4.12)$$

Here again the problem is solved by the method of Erdogan and Gupta and the results for the variation of $K^{(1)}/K_0^{(1)}$, $-K^{(2)}/K_0^{(2)}$ and W/W_0 with a_1/b for several values of a_2/b are shown in Figures 6 and 7 respectively.

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